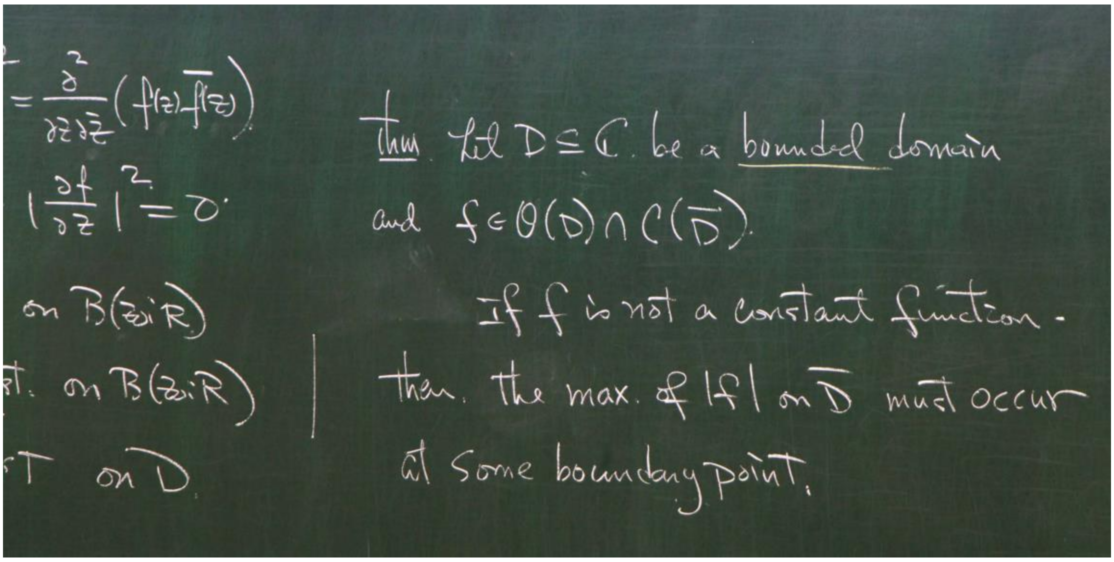
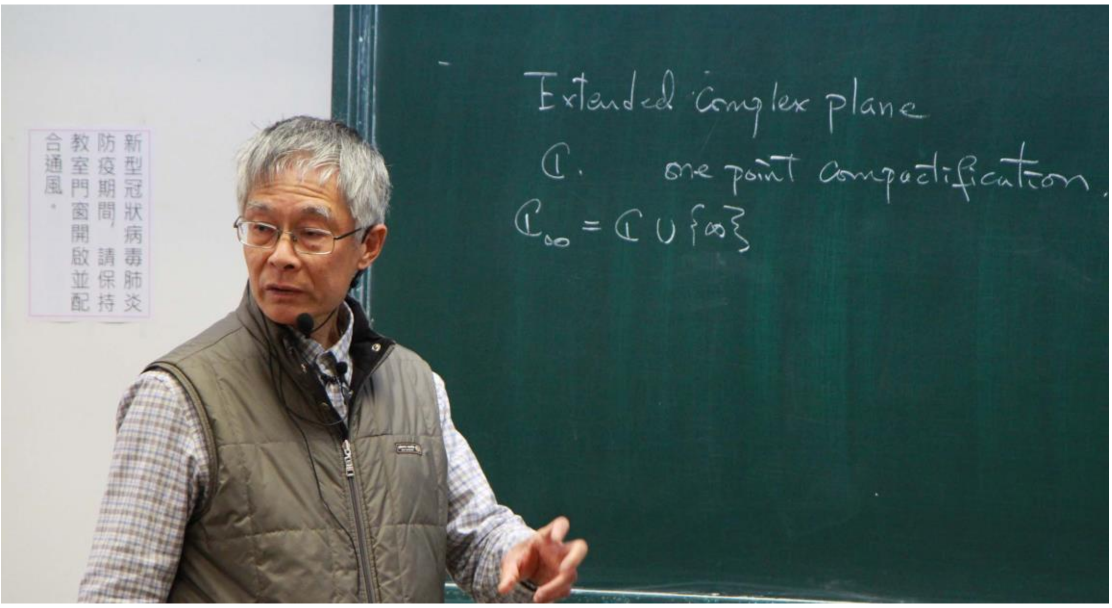
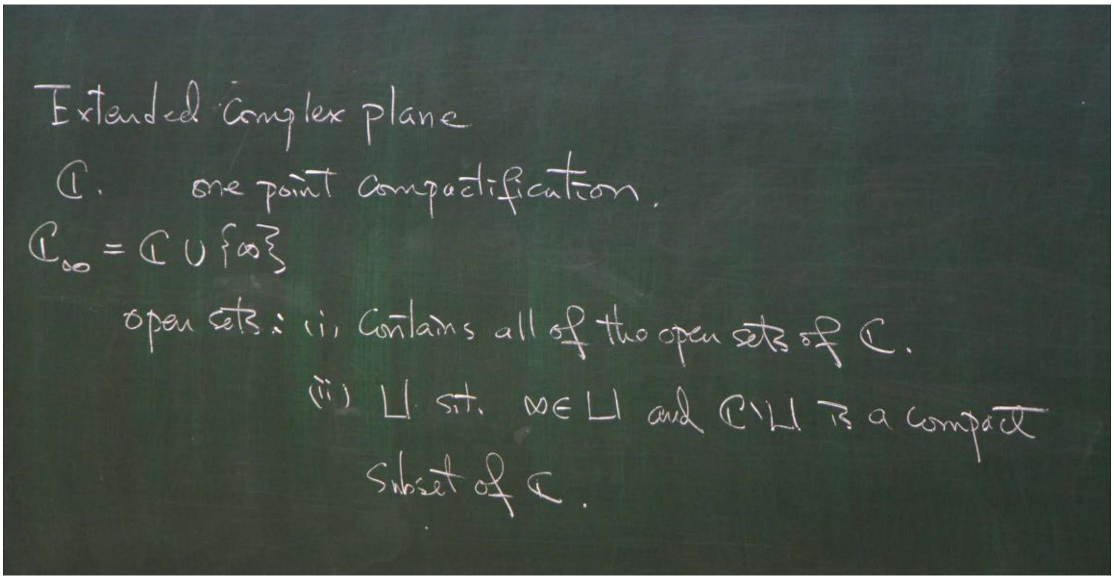
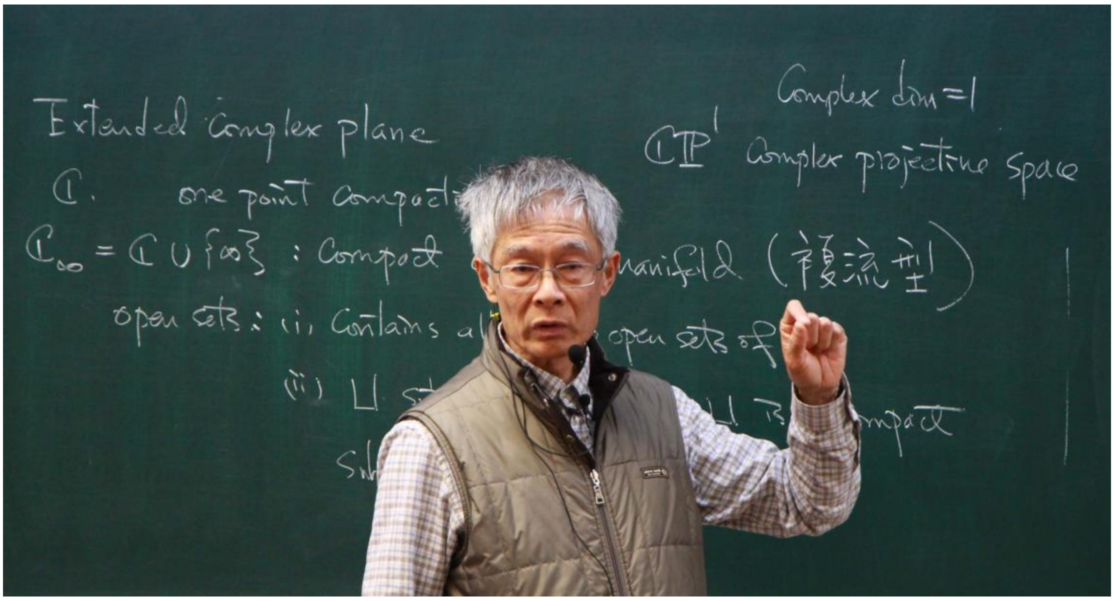
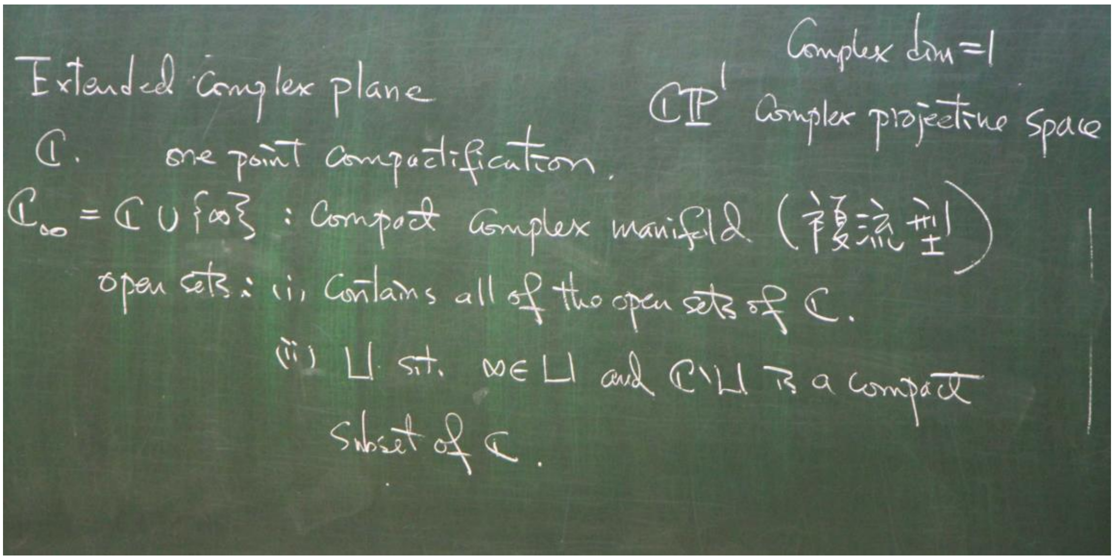
【10920 程守慶教授複變數函數論 / 第 5 堂版書】




Extended complex plane
$\mathbb{C}$. one point compactification.
$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$
open sets: (i) Contains all of the open sets of $\mathbb{C}$.
(ii) $U$ s.t. $\infty \in U$ and $\mathbb{C} \setminus U$ is a compact
subset of $\mathbb{C}$.


Complex dim = 1
Extended complex plane
$\mathbb{CP}^1$ Complex projective space
$\mathbb{C}$. one point compact
$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ : Compact
manifold (複流型)
open sets: (i) Contains al
open sets of
(ii) $U$
mpact


Complex dim = 1
Extended complex plane
$\mathbb{CP}^1$ Complex projective space
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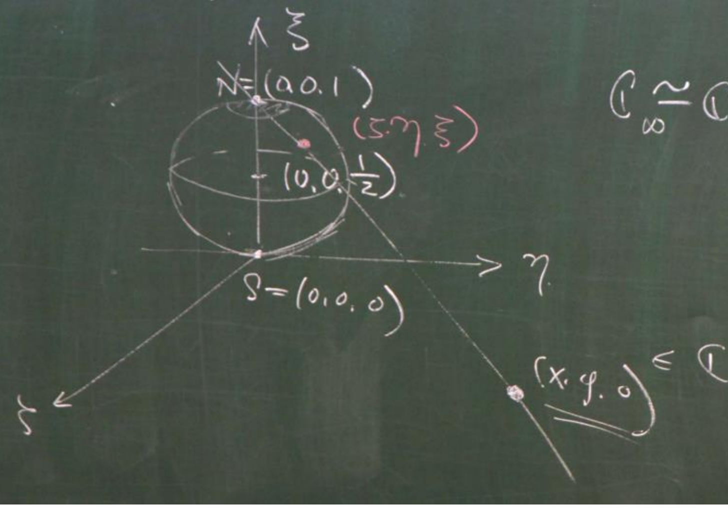
Riemann sphere $S^2$




$\mathbb{C}_\infty \simeq \mathbb{CP}^1 \simeq S^2$

立体投影,
Stereographic projection

Riemann's theorem on removable singularity.

$D \subseteq \mathbb{C}$. domain.

$z_0 \in D$

$f \in \mathcal{O}(D \setminus \{z_0\})$ and $|f|$ is bounded on $D \setminus \{z_0\}$.

Then $f$ can be redifined at $z_0$. s.t.
the function, still denoted by $f$, is holomorphic in $D$. i.e. $f \in \mathcal{O}(D)$

$\{z_0\}$.

...omorphic

pf. Consider

$|f(z)| \le M$

$$g(z) = (z-z_0)^2 f(z) \quad \text{on } D \setminus \{z_0\}$$

✓ $g(z_0) = 0$

$\therefore\ g \in C(D)$.

$$\lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{(z-z_0)^2 f(z)}{z-z_0} = \lim_{z\to z_0} (z-z_0) f(z) = 0$$

✓ $g'(z_0)$ ''

$\therefore\ g \in \overset{?}{\mathcal{O}}(D)$


$|f(z)| \le M$

$\therefore\ g \in \overset{?}{\mathcal{O}}(D)$

$)^2 f(z)$ on $D \setminus \{z_0\}$

$= \lim_{z\to z_0} \frac{(z-z_0)^2 f(z)}{z-z_0} = \lim_{z\to z_0} (z-z_0) f(z) = 0$


$$\therefore\ g(z) = \sum_{k=2}^{\infty} a_k (z-z_0)^k = (z-z_0)^2 \left( \sum_{m=0}^{\infty} a_{m+2} (z-z_0)^m \right)$$

$$= (z-z_0)^2 \varphi(z)$$

$\varphi \in \mathcal{O}(D)$.

$\therefore\ f(z) = \varphi(z)$ on $D \setminus \{z_0\}$

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \varphi(z) = \varphi(z_0).$$

$|f(z)| \le M$

$\ldots)^2 f(z)$ on $D\setminus\{z_0\}$

$\therefore\ g \overset{?}{\in} \mathcal{O}(D)$

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$$= (z-z_0)^2 \varphi(z)$$

$\varphi \in \mathcal{O}(D)$.

$\therefore\ f(z) = \varphi(z)$ on $D\setminus\{z_0\}$

$(z-z_0) f(z) = 0$

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \varphi(z) = \varphi(z_0).$$

If we redefine $f$ at $z_0$ to be $\varphi(z_0)$.

Then $f = \varphi \in \mathcal{O}(D)$.

(II)

$|f(z)| \to \infty$ as $z \to z_0$

i.e. $|f(z)| \geq M > 0$ when $z \in B(z_0;\delta) \setminus \{z_0\}$

Consider

$$g(z) = \frac{1}{f(z)} \quad \therefore \ |g(z)| \leq \frac{1}{M} \quad \text{on}$$

$$\therefore \lim_{z \to z_0} g(z) = 0.$$


(II)

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$$\varphi \in \mathcal{O}(B(z_0;\delta)) \qquad \varphi(z_0) = a_{k_0} \neq 0.$$

$$g(z) = (z-z_0)^{k_0} \varphi(z)$$

$$\therefore g(z) = \sum_{k=k_0}^{\infty} a_k (z-z_0)^k = (z-z_0)^{k_0} \left( \sum_{m=0}^{\infty} a_{k_0+m} (z-z_0)^m \right)$$

$k_0$: smallest index s.t. $a_{k_0} \neq 0$

$$g(z) = \frac{1}{f(z)} = (z-z_0)^{k_0}\varphi(z) \qquad \varphi(z_0) \neq 0. \qquad h(z_0) = c_0 \neq 0.$$

On $z \in B(z_0;\delta)\setminus\{z_0\}$

$$f(z) = \frac{1}{(z-z_0)^{k_0}\varphi(z)} = \frac{1}{(z-z_0)^{k_0}}\, h(z) = \frac{1}{(z-z_0)^{k_0}}\left(c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots\right)$$

$$\therefore\ f(z) = \underbrace{\boxed{\frac{c_0}{(z-z_0)^{k_0}} + \frac{c_1}{(z-z_0)^{k_0-1}} + \cdots + \frac{c_{k_0-1}}{z-z_0}}}_{= I_{z_0}} + \overbrace{\boxed{c_{k_0} + c_{k_0+1}(z-z_0) + \cdots + c_{k_0+m}(z-z_0)^m + \cdots}}^{h_{z_0}}$$

$$f(z) = I_{z_0} + h_{z_0}$$

↑ Singular part

↖ holomorphic part

$$f(z) - I_{z_0} \in \mathcal{O}$$

$k_0$ is called the order of the pole.

A function $f$ is called a meromorphic function on $D$ if $f$ is holomorphic except for poles.

e singularity.

ounded on $D \setminus \{z_0\}$

t $z_0$. s.t.

g $f$. is holomorphic

亜純函数

A function $f$ is called a <u>meromorphic function</u> on $D$ if $f$ is holomorphic except for poles.

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$$f(z) = I_{z_0} + h_{z_0}$$

$I_{z_0}$: singular part

$h_{z_0}$: holomorphic part

$$f(z) - I_{z_0} \in \mathcal{O}$$

$k_0$ is called the order of the pole

$k_0 = 1$. we call it <u>simple pole</u>.

$$f(z) = \frac{1}{z}$$

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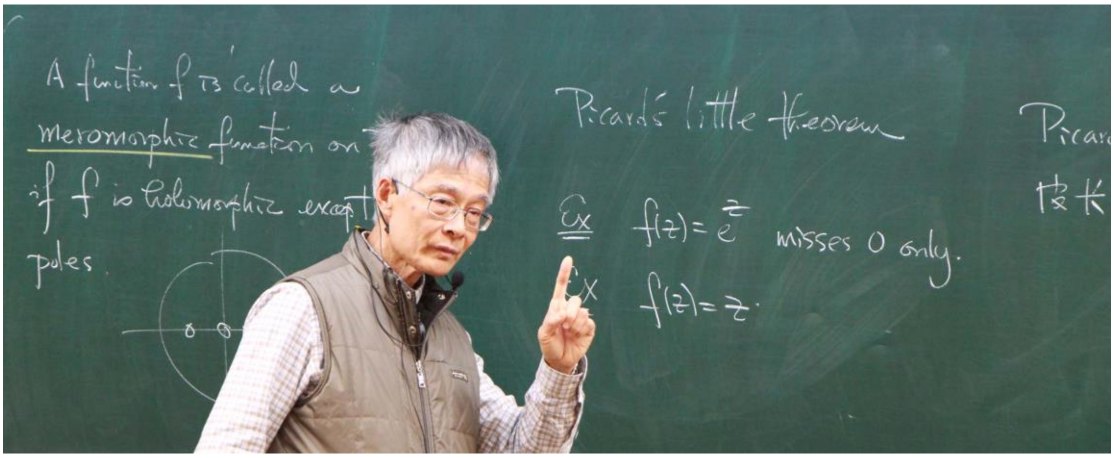
Picard's little theorem

<u>Ex</u> $f(z) = e^{z}$

Picard's great theorem

皮卡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Ex $f(z)=e^{z}$ misses 0 only.
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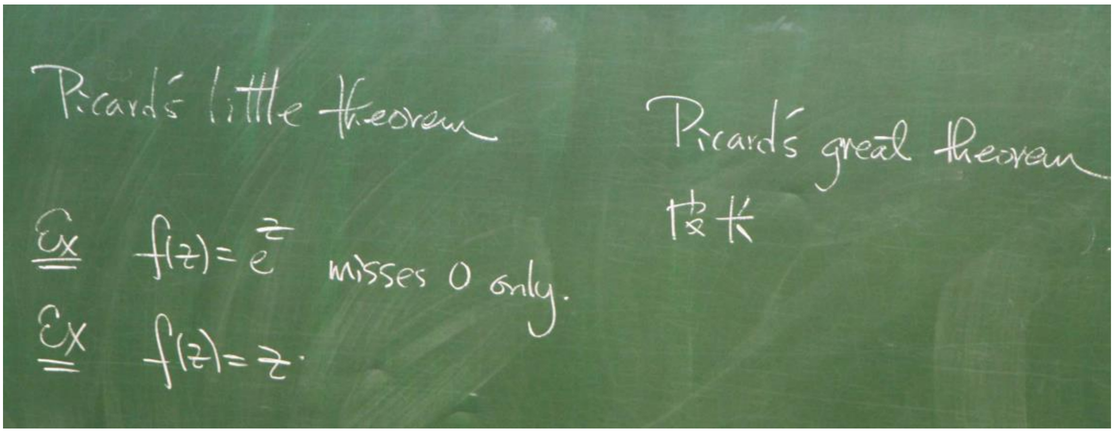
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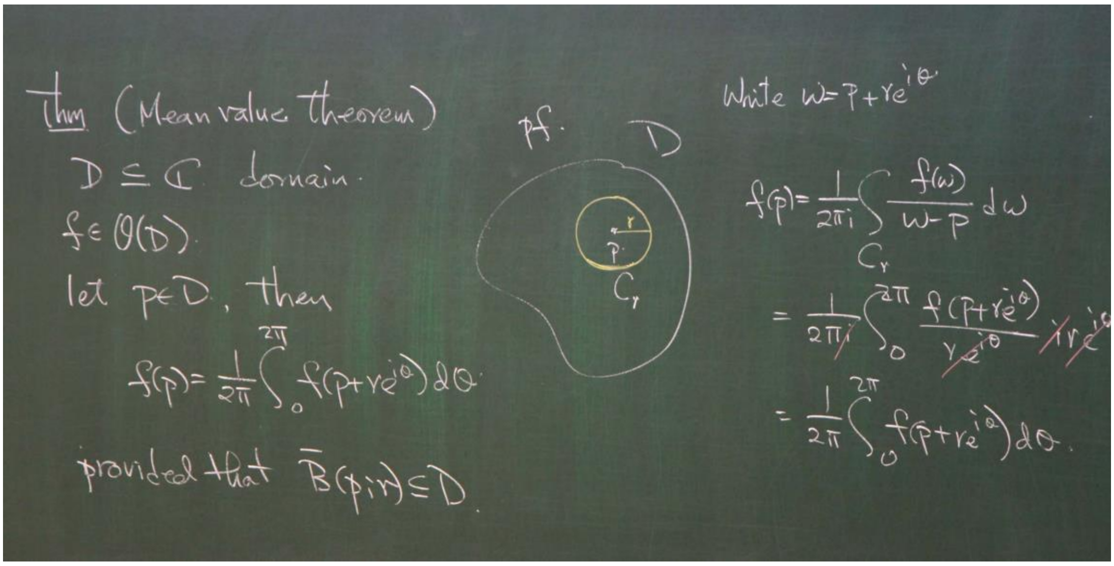
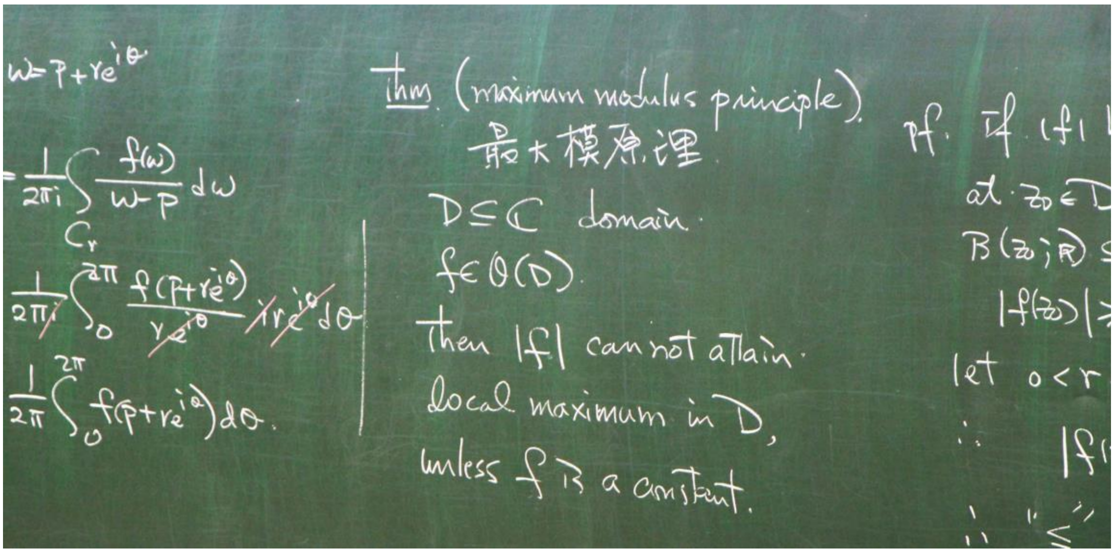
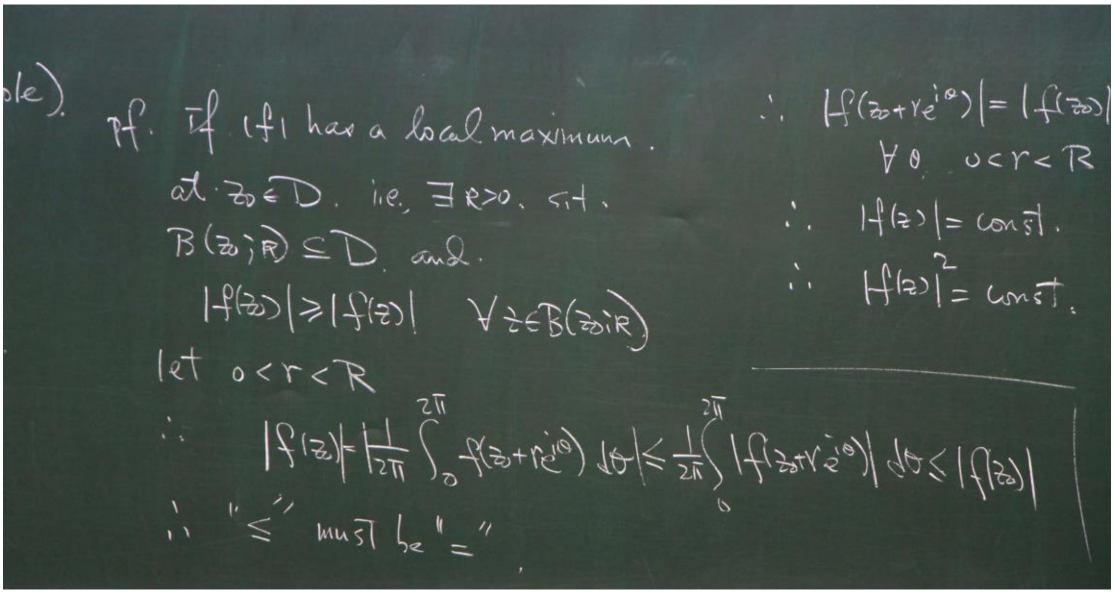
Picard's great theorem
长


thm (Mean value theorem)
$D \subseteq \mathbb{C}$ domain.
$f \in \mathcal{O}(D)$.
let $p \in D$, then
$f(p)=\frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\,d\theta$.
provided that $\overline{B}(p;r) \subseteq D$.
pf.
D
r
p
$C_r$
Write $w=p+re^{i\theta}$
$f(p)=\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-p}\,dw$
$=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(p+re^{i\theta})}{re^{i\theta}}\,ire^{i\theta}$
$=\frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\,d\theta$.


$w = p + re^{i\theta}$
$= \frac{1}{2\pi i}\int_{C_r} \frac{f(w)}{w-p}\,dw$
$= \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(p+re^{i\theta})}{\cancel{re^{i\theta}}}\,\cancel{ire^{i\theta}}\,d\theta$
$= \frac{1}{2\pi}\int_0^{2\pi} f(p+re^{i\theta})\,d\theta.$
thm. (maximum modulus principle).
最大模原理
$D \subseteq \mathbb{C}$ domain.
$f \in \mathcal{O}(D)$.
Then $|f|$ can not attain local maximum in $D$, unless $f$ is a constant.


pf. If $|f|$ has a local maximum at $z_0 \in D$. i.e., $\exists R>0$, s.t. $B(z_0;R) \subseteq D$ and
$|f(z_0)| \geq |f(z)| \quad \forall z \in B(z_0;R)$
let $0<r<R$
$\therefore \quad |f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta \leq |f(z_0)|$
$\therefore$ "$\leq$" must be "$=$".
$\therefore \quad |f(z_0+re^{i\theta})| = |f(z_0)| \quad \forall\theta, \ 0<r<R$
$\therefore \quad |f(z)| = \text{const}.$
$\therefore \quad |f(z)|^2 = \text{const}.$

$\therefore\ |f(z_0 + re^{i\theta})| = |f(z_0)|$
$\quad \forall \theta,\ 0 < r < R$

$\therefore\ |f(z)| = \text{const.} \quad z \in B(z_0; R)$

$\therefore\ |f(z)|^2 = \text{const.} \quad$ "

$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta \le |f(z_0)|$

$\therefore\ \frac{\partial^2}{\partial z \partial \bar{z}} |f(z)|^2 = \frac{\partial^2}{\partial z \partial \bar{z}} \left( f(z) \overline{f(z)} \right)$
$= \left| \frac{\partial f}{\partial z} \right|^2 = 0.$

$\therefore\ f'(z) = 0$, on $B(z_0; R)$

$\therefore\ f(z) = \text{const.}$ on $B(z_0; R)$

$\therefore\ f(z) = \text{const}$ on $D$.

$\frac{\partial^2}{\partial z \partial \bar{z}} \left( f(z) \overline{f(z)} \right)$
$\left| \frac{\partial f}{\partial z} \right|^2 = 0.$
$B(z_0; R)$
on $B(z_0; R)$
on $D$.

<u>thm</u>. Let $D \subseteq \mathbb{C}$ be a <u>bounded</u> domain and $f \in \mathcal{O}(D) \cap C(\overline{D})$.

If $f$ is not a constant function, then the max. of $|f|$ on $\overline{D}$ must occur at some boundary point.